CSCI 1377
Tools for Thought

Notation I
Compacting Concepts

"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the [human] race."

— Alfred North Whitehead, *An Introduction to Mathematics* (1911)

notation (n.)

a collection of related symbols that are each given an arbitrary meaning, created to facilitate structured communication within a domain

Script

Hello world

こんにちは世界

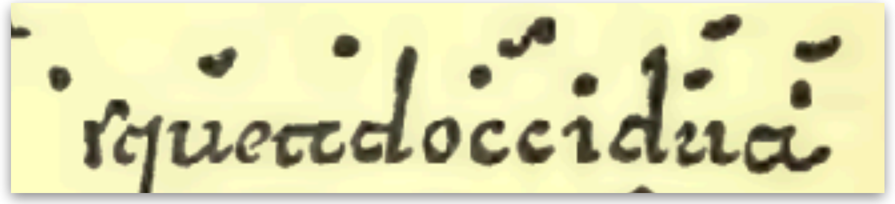
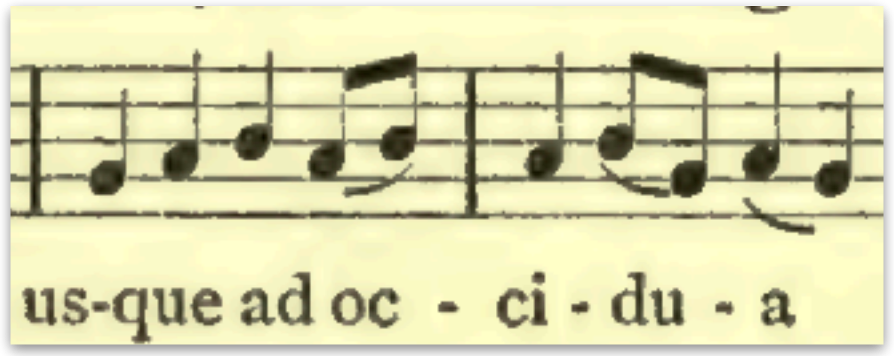
Numerals

2026

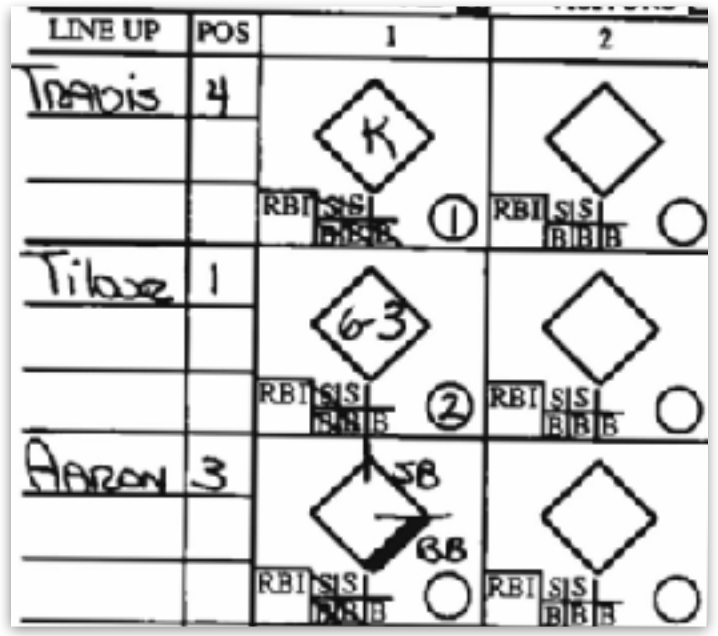
二千二十六

MMXXVI

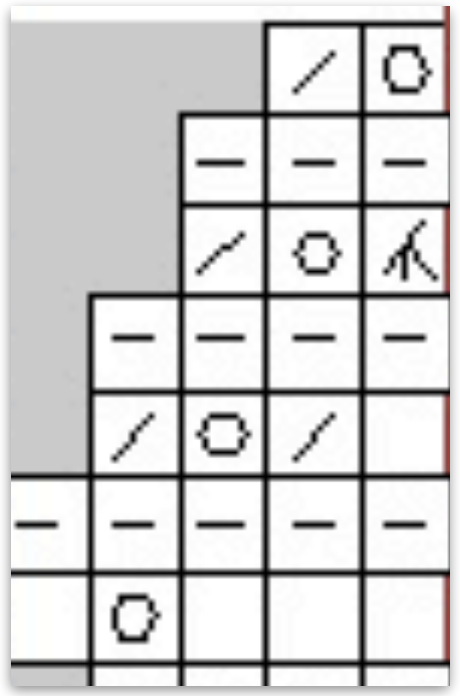
Music



Baseball



Knitting



This unit:

- Where do notations come from?
- What makes a good notation?
- What is the role of notation in communicating with computers?

Mathematics used to be prosaic

$$\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

$$x^2 + 21 = 10x \qquad x^2 + c = bx$$

"Halve the number of the roots. It is 5. Multiply this by itself and the product is 25. Subtract from this the 21 added to the square term and the remainder is 4. Extract its square root, 2, and subtract this from half the number of roots, 5. There remains 3. This is the root you wanted, whose square is 9. Alternatively, you may add the square root to half the number of roots and the sum is 7. This is then the root you wanted and the square is 49."

$$10 \div 2 = 5 \qquad n_1 = b \div 2$$

$$5^2 = 25 \qquad n_2 = n_1^2$$

$$25 - 21 = 4 \qquad n_3 = n_2 - c$$

$$\sqrt{4} = 2 \qquad n_4 = \sqrt{n_3}$$

$$5 - 2 = 3 \qquad n_5 = n_1 - n_4$$

https://mathshistory.st-andrews.ac.uk/Extras/Al-Khwarizmi_quadratics/

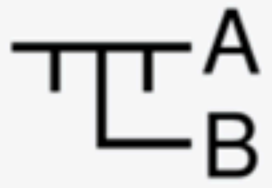
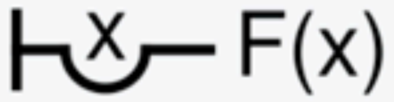
Begriffsschrift: "formula language for pure thought"

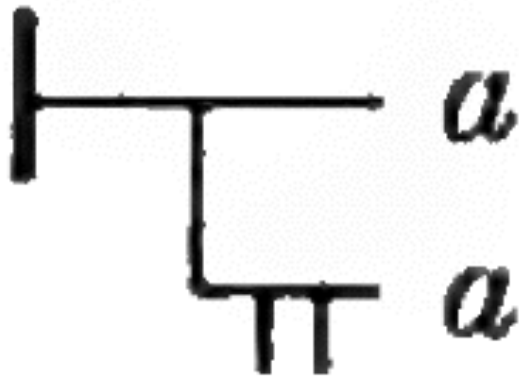
"In attempting to comply with this requirement [of rigor] in the strictest possible way I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated.

Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. [...] But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others. This ideography, likewise, is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others."

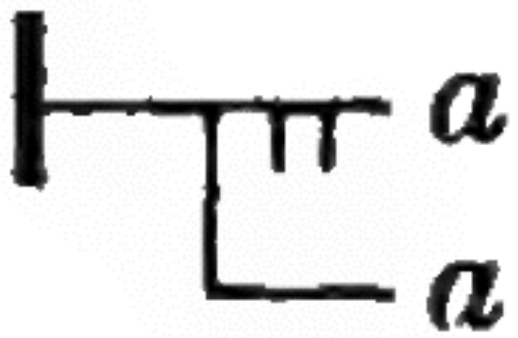
Gottlob Frege. *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought.* 1879

Begriffsschrift: “formula language for pure thought”

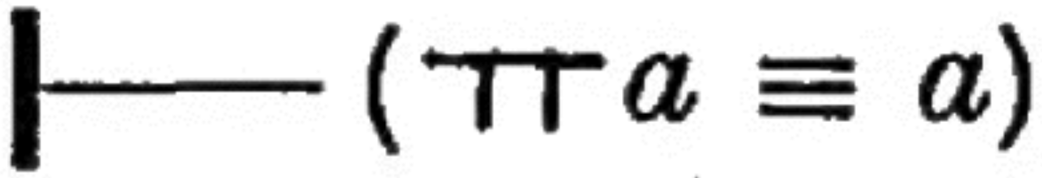
Basic concept	Frege's notation	The diagram shows (in modern notation)	Modern notation
Judging	$\vdash A, \Vdash A$		$p(A) = 1,$ $p(A) = i \vdash A, \Vdash A$
Negation		basic	$\neg A$
Material conditional		basic	$B \to A$
Logical conjunction		$\neg(B \to \neg A)$	$A \wedge B$
Logical disjunction		$\neg B \to A$	$A \vee B$
Universal quantification		basic	$\forall x\, F(x)$
Existential quantification		$\neg \forall x\, \neg F(x)$	$\exists x\, F(x)$
Material equivalence	$A \equiv B$		$A \leftrightarrow B$
Identity	$A \equiv B$		$A = B$



$$\neg\neg a \implies a$$



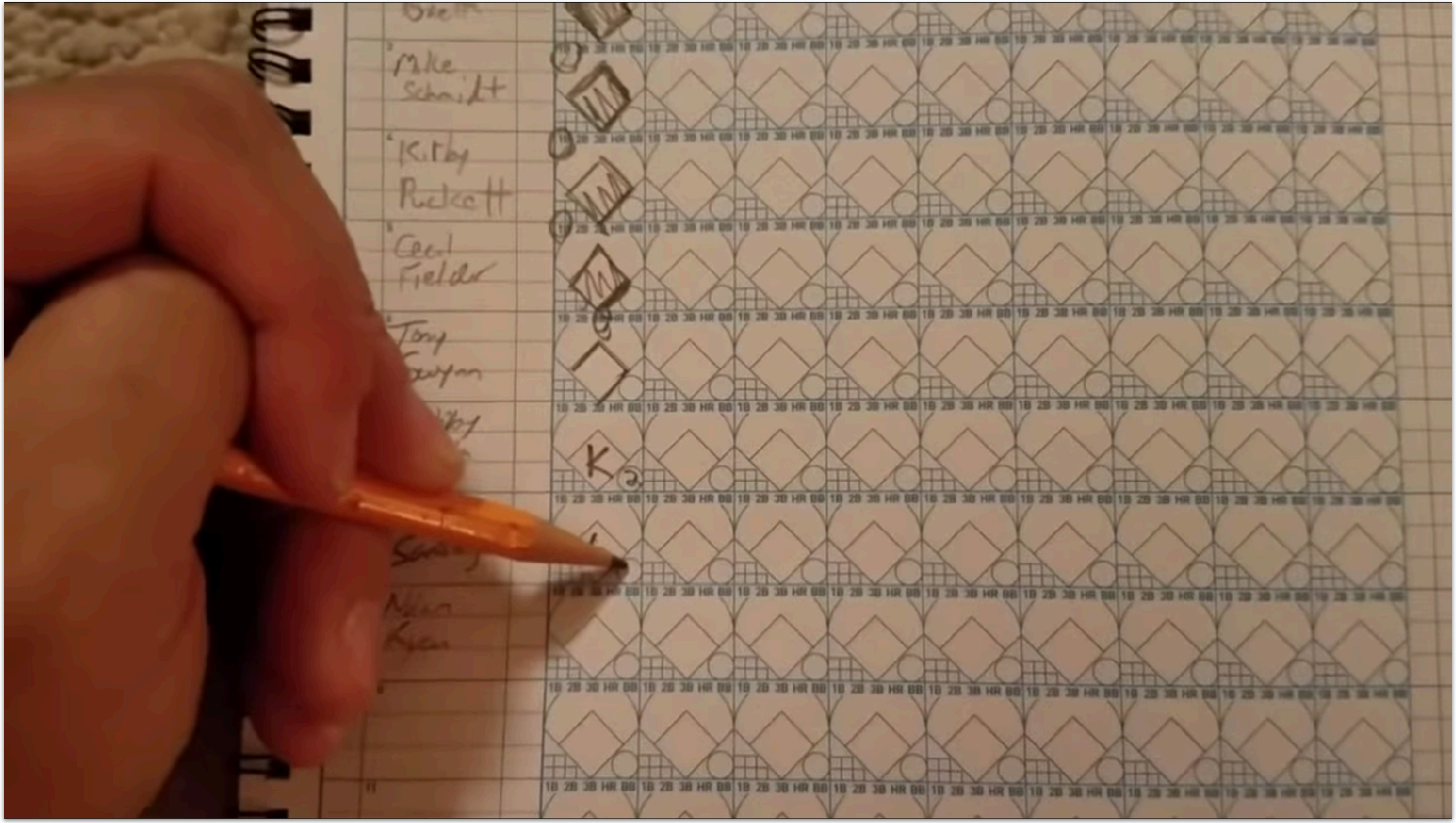
$$a \implies \neg\neg a$$



$$a \iff \neg\neg a$$

Gottlob Frege. *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought.* 1879

Modern notations



youtu.be/Ei6DGH_Y6Q8

MATCH BETWEEN	AUSTRALIA v ENGLAND	PLAYED AT	WOOLLOONGABBA	COMPETITION	ASHES		
1st or 2nd INNINGS of	AUSTRALIA (1ST)	TOSS WON BY	AUSTRALIA	DATE	25-NOV-1994	GRADE TEST	ROUND NO. 1

OVR	RUNS	OVR	RUNS
1	8	51	149
2	14	52	157
3	18	53	160
4	26	54	176
5	26	55	182
6	27	56	186
7	27	57	190
8	31	58	194 T
9	37	59	196
10	41	60	201
11	44	61	212
12	45	62	216
13	45	63	219
14	45	64	228
15	50	65	234
16	53	66	240
17	53	67	245
18	58	68	250
19	61	69	254
20	61	70	260
21	62	71	262
22	68	72	266
23	70	73	276
24	75	74	277
25	77	75	288
26	81	76	293
27	84	77	299
28	84	78	299
29	87 L	79	308
30	92	80	311
31	94	81	313
32	98	82	313
33	99	83	313
34	107	84	319
35	112	85	324
36	117	86	326
37	117	87	326
38	118	88	326
39	121	89	327
40	124	90	329 S
41	125	91	329
42	126	92	332
43	127	93	333
44	127	94	347
45	129	95	352
46	134	96	356
47	134	97	356
48	137	98	358
49	139	99	358
50	144	100	365

TIME IN	TIME OUT	MINS	BATSMAN	HOW OUT	BOWLER	TOTAL
11.00	5.23	323	1. M.J. SLATER	c GATTING	GOOCH	176 (244)
11.00	1.53	133	2. M.A. TAYLOR	RUN OUT		59 (110)
1.55	2.29	34	3. D.C. BOON	BOWLED	GOUGH	3 (24)
2.30	1.47	317	4. M.E. WAUGH	c STEWART	GOUGH	140 (215)
5.24	5.49	35	5. M.G. BEVAN	c HICK	GOUGH	7 (18)
5.50	11.24	34	6. S.K. WARNE	c RHODES	GOUGH	2 (25)
11.25	11.58	33	7. S.R. WAUGH	c HICK	DeFREITAS	19 (30)
12.01	12.34	33	8. I.A. HEALY	c HICK	DeFREITAS	7 (25)
12.36	12.54	18	9. C.J. McDERMOTT	c GOUGH	McCAGUE	2 (7)
12.56		20	10. T.B.A. MAY	NOT	OUT	3 (21)
1.48	1.56	8	11. G.D. McGRATH	c GOUGH	McCAGUE	0 (5)

EXTRAS		
BYES	32	5
LEG BYES	11	2
WIDES		–
NO BALLS	1	1
TOTAL SCORE		426

FALL OF WICKETS	1 for 99	2 for 126	3 for 308	4 for 326	5 for 352	6 for 379	7 for 407	8 for 419	9 for 425	10 for 426
OUTGOING BATSMAN	TAYLOR	BOON	SLATER	BEVAN	WARNE	S.WAUGH	HEALY	McDERMOTT	M.WAUGH	McGRATH
NOT OUT BATSMAN	SLATER	SLATER	M.WAUGH	M.WAUGH	M.WAUGH	M.WAUGH	M.WAUGH	M.WAUGH	MAY	MAY

BOWLING ANALYSIS

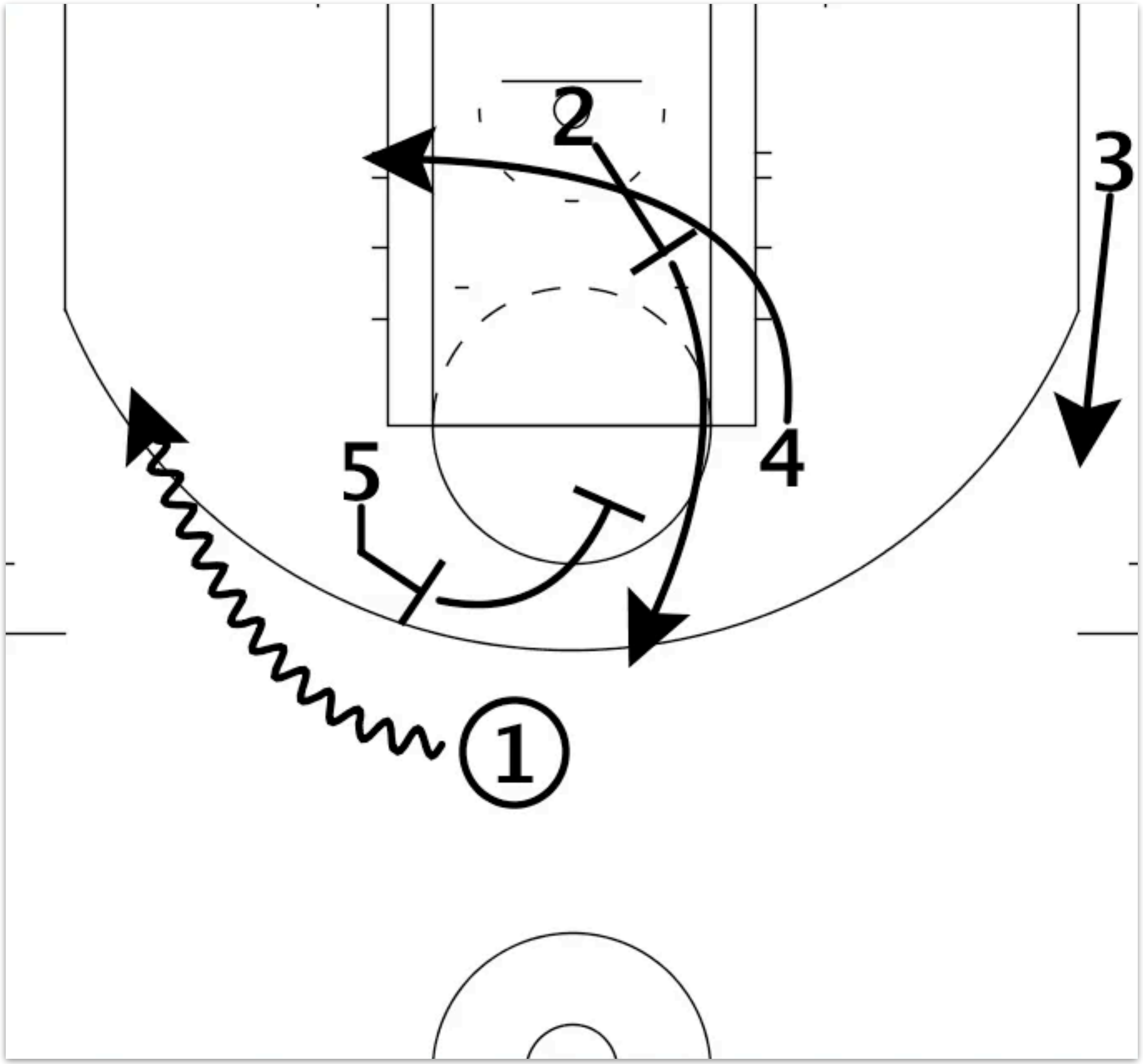
BOWLER	OVER	MDNS	WIDES	NO BALLS	WKTS	RUNS
1. DeFREITAS	31	8			2	102
2. McCAGUE	19.2	4			2	96
3. GOUGH	32	7			4	107
4. TUFNELL	25	3			–	72
5. HICK	4	–			–	22
6. GOOCH	9	2			1	20
7. GOUGH						
8. TUFNELL						

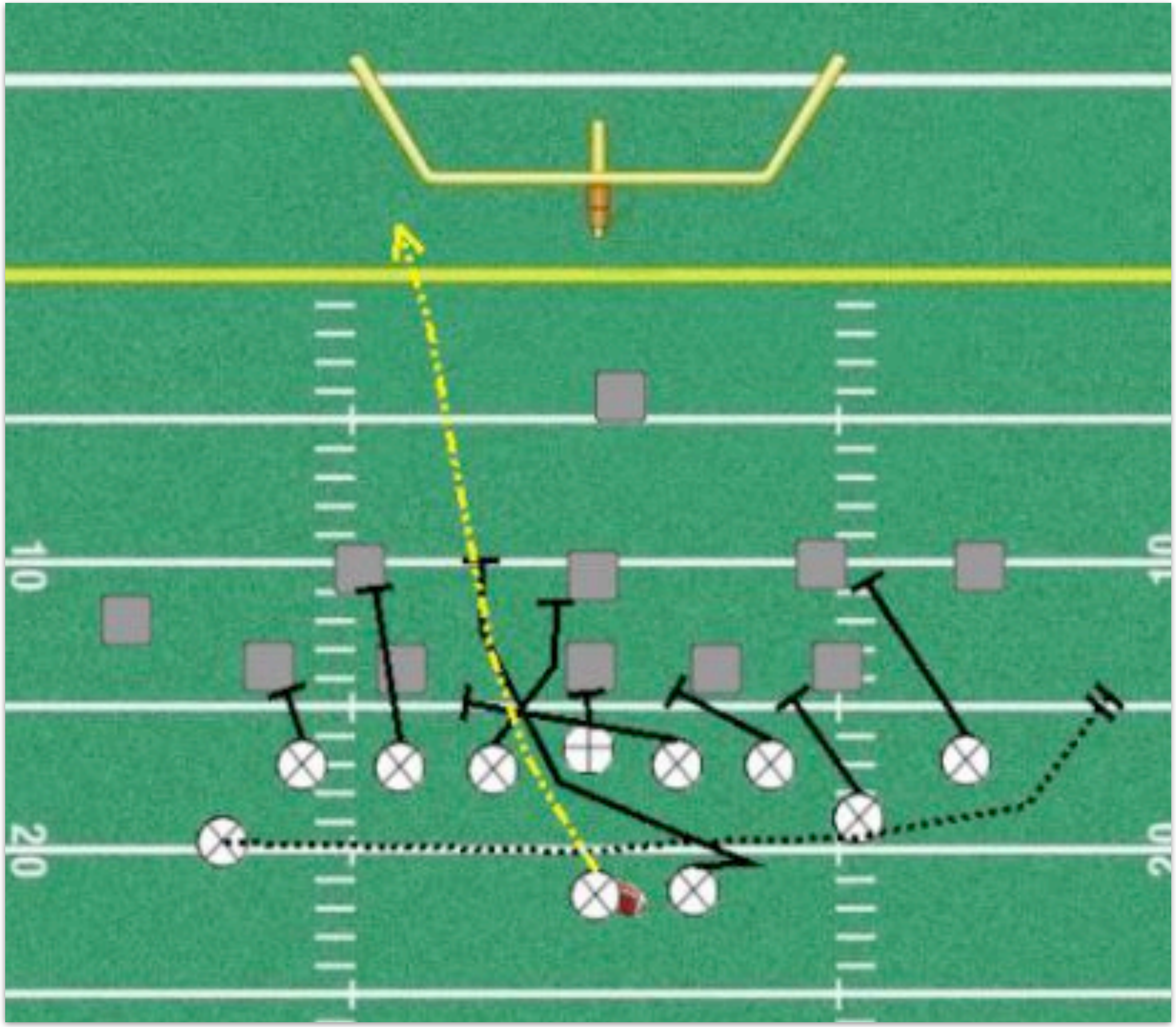
NOTES: HICK 4TH OVER 4TH BALL 4 OVERTHROWS

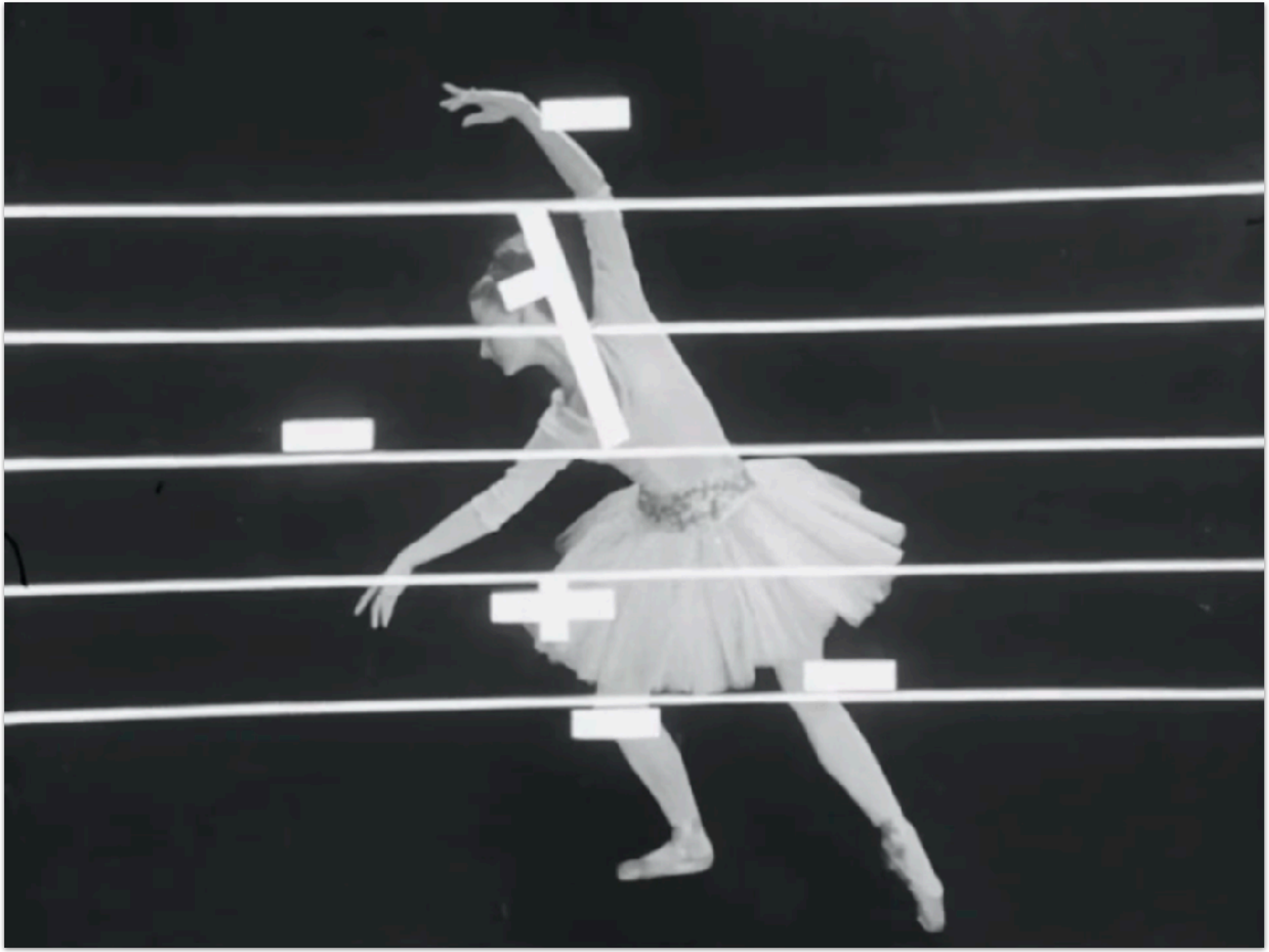
UMPIRES: RANDALL (A) & MITCHLEY (SA)

RESULT:

Play notation indicates concurrent motion







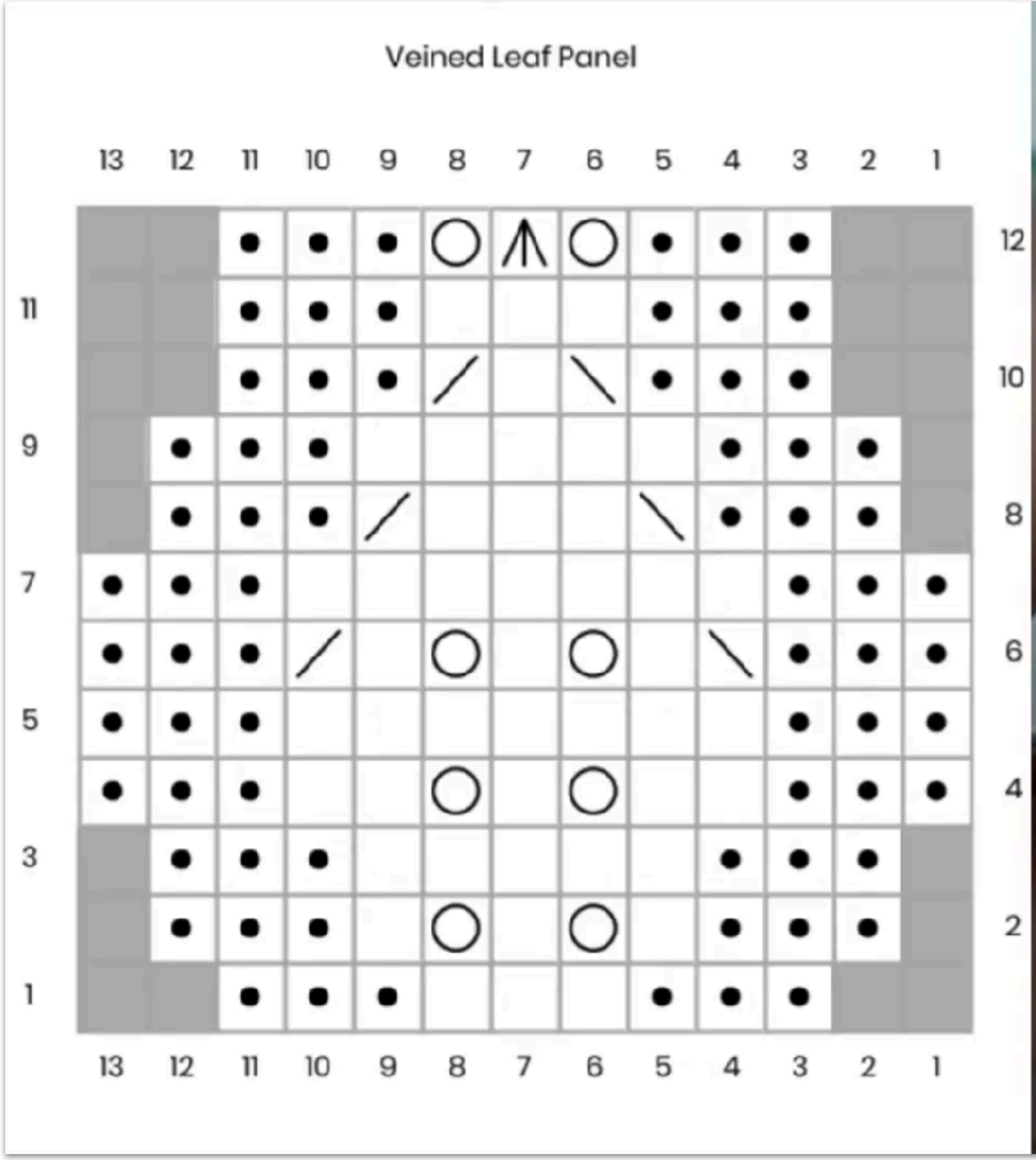
youtu.be/g6SuQTNGM8c

Dance notation is not as prevalent as in music

Today dance notation is arcane, and mostly inessential to the art of dance. Even the two most prevalent systems, Laban and Benesh, don't enjoy wide literacy among dancers. [...]

The systems, each of which may hold some slight improvement over its predecessor, are so difficult to decipher, even to initial mastery of their alphabet, that when students approach the problem of putting the letters together, or finally fitting the phrases to music, they feel triumphant if they can decipher even a single short solo *enchaînement.* An analysis of style is not attempted, and the problem of combining solo variations with a *corps de ballet* to provide a chart of an *entire* ballet movement reduces the complexity of the problem to the apoplectic.

— Anna Heyward, "How to Write a Dance" (2015)



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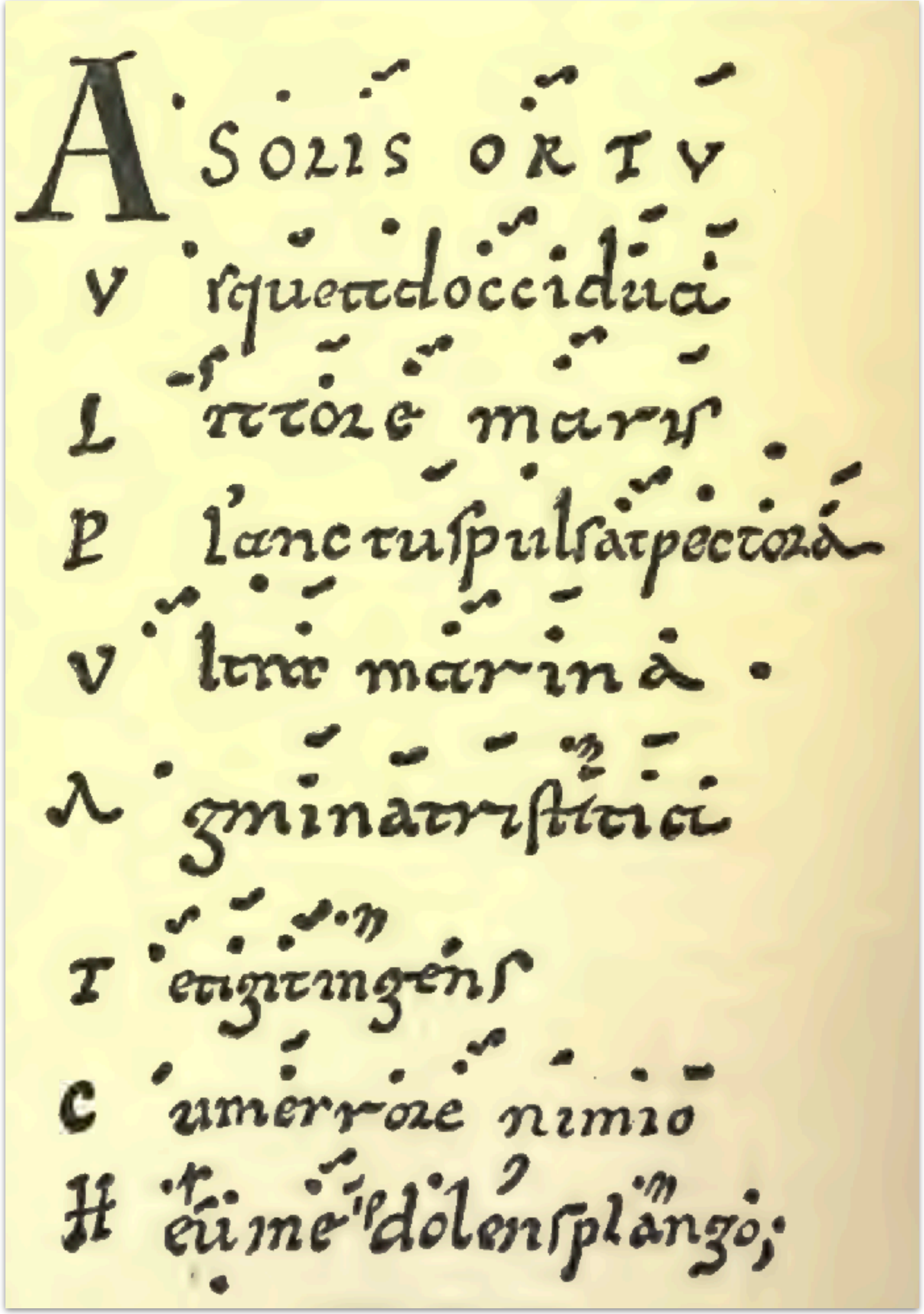
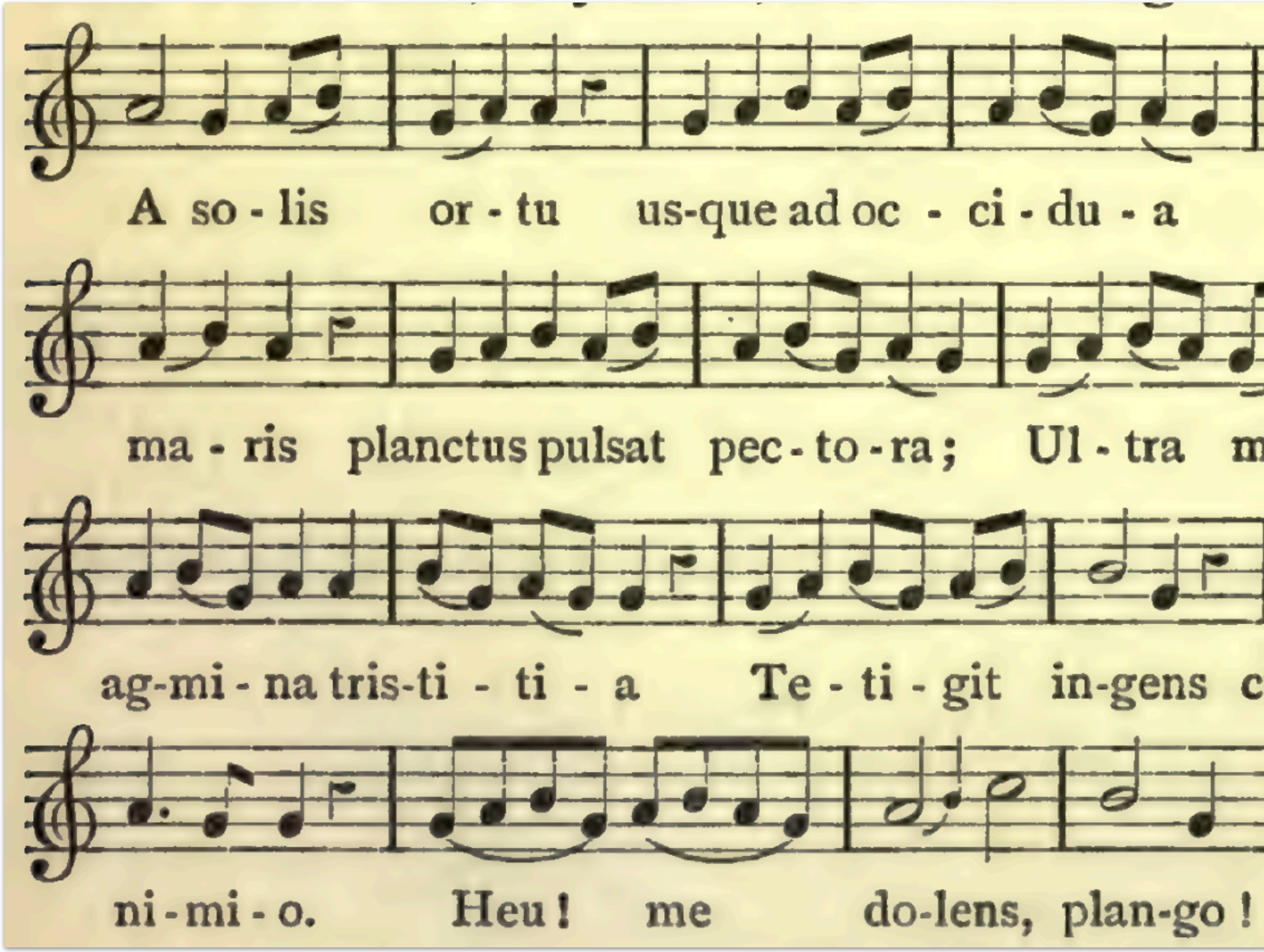
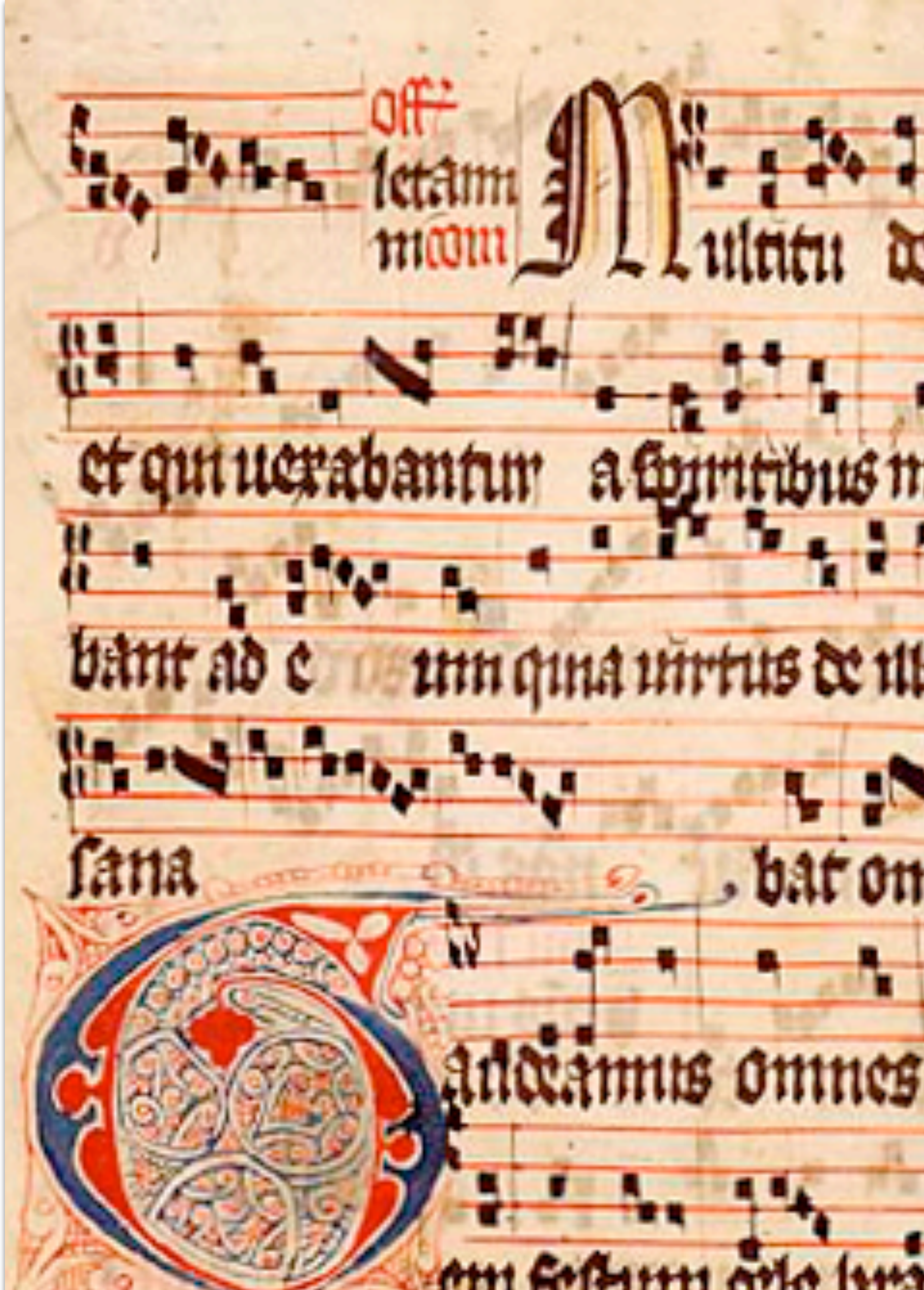


Williams. *The Story of Notation.* 1903

Medieval music was encoded on a staff with neumes

> "Any improvement in a universally accepted notation comes very slowly, not as the result of one man's inspiration, but by a consensus of opinion that such and such a detail requires to be, and can be, improved. The change from square and lozenge notes, for instance, to round ones took some centuries to complete; it was not hte result of someone's gugestion, but a requirement of rapid writing."

Williams. *The Story of Notation.* 1903

via Wikimedia Commons

Mathematical notation

Modern math notation is the result of centuries of organic innovation and spread

Diophantus, 3rd century CE

From v. 10: $\frac{\iota\beta}{\iota\zeta} = \frac{17}{12}$

From v. 8, Lemma: $\bar{\beta}\angle' s' = 2\,\frac{1}{2}\,\frac{1}{8}$

From iv. 3: $s \chi \bar{\eta} = \frac{8}{x}$

From iv. 15: $\Delta^{Y} \chi\, \overline{\sigma\nu} = \frac{250}{x^2}$

From vi. 12: $\Delta^{Y}\bar{\xi}\mathring{M}\overline{,\beta\phi\kappa}$ ἐν γορίῳ $\Delta^{Y}\Delta\bar{\alpha}\mathring{M}$ ϡ $\pitchfork \Delta^{Y}\bar{\xi}$

$= (60x^2+2{,}520)/(x^4+900-60x^2)$.

Luca Pacioli, 1494

℞.200. for $\sqrt{200}$

℞.*cuba. de* .64. for $\sqrt[3]{64}$

℞.*relato*. for fifth root

℞ ℞ ℞. *cuba*. for seventh root

℞ .6.$\tilde{m}$.℞.2. for $\sqrt{6}-\sqrt{2}$

Francesco Ghaligai, 1552

$\frac{1}{4}$ □ *di* □ $\tilde{m}$ $\frac{1}{4}$ *di* □ — 1 □ $\quad \frac{1}{4}x^4 - \frac{1}{4}x^2 = x^2$

$\frac{1}{4}$ □ *di* □ — $1\frac{1}{4}$ *di* □ $\quad \frac{1}{4}x^4 = 1\frac{1}{4}x^2$

$\frac{1}{4}$ □ — — — — $1\frac{1}{4}$ $\tilde{n}$ $\quad \frac{1}{4}x^2 = 1\frac{1}{4}$

Cajori. "A History of Mathematical Notation". 1929

Dozens of ways to notate a dot product

$$c = \sum_{i=1}^{n} u_i v_i$$

$$c = u_1 v_1 + \ldots + u_n v_n$$

$$c = u_1 v_1 + \ldots + u_n v_n$$

$$c = \langle u, v \rangle$$

$$c = \eta(u, v)$$

$$c = \langle u \,|\, v \rangle$$

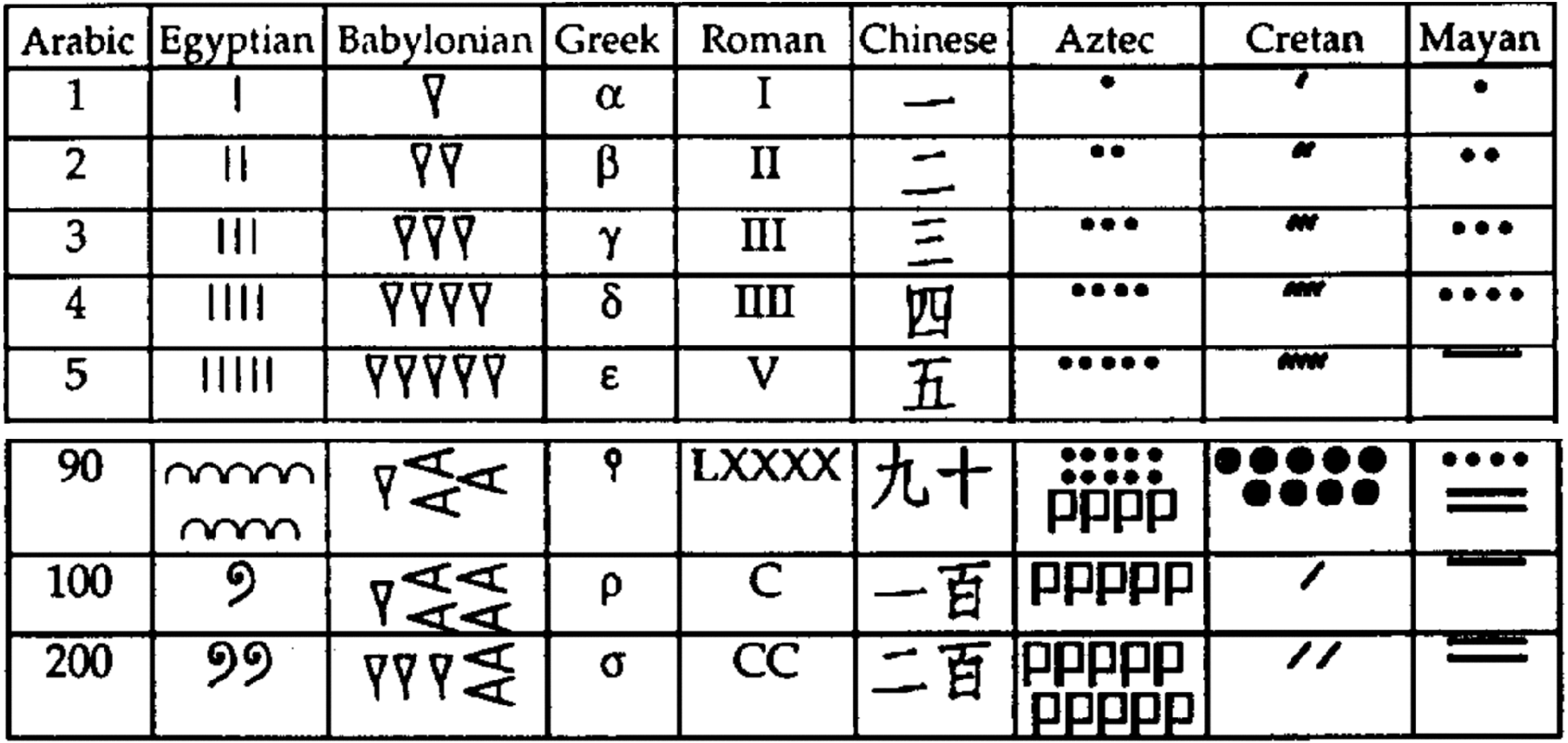
$$c = u_i v_i$$

$$c = \det(u \wedge {*v})$$

$$c = \frac{1}{2}\{u, v\}$$

https://mathoverflow.net/questions/366070/what-are-the-benefits-of-writing-vector-inner-products-as-langle-u-v-rangle/366118#366118

Numerals have been invented many times



Zhang and Norman. “A Representational Analysis of Numeration Systems” 1995

Numerals encode power and base in $1 \times 1\text{D}$

Systems	Example (447)	Base	Base Dimension	Power Dimension
Abstract	$\Sigma\, a_i x^i$	x	a_i	x^i
Arabic	447	10	a_i = shape	x^i = position
	$4\times10^2+4\times10^1+7\times10^0$		0, 1, 2, ..., 9	... 10^2 10^1 10^0
Egyptian	9999∩∩∩∩|||||||	10	a_i = quantity	x^i = shape
	$4\times10^2+4\times10^1+7\times10^0$		Quantities of |'s, ∩'s, 9's, etc.	| ∩ 9 ... 10^0 10^1 10^2 ...
Cretan	////●●●●•••••••	10	a_i = quantity	x^i = shape
	$4\times10^2+4\times10^1+7\times10^0$		Quantities of •'s, ●'s, /'s, etc.	• ● / ... 10^0 10^1 10^2 ...

Zhang and Norman. "A Representational Analysis of Numeration Systems" 1995

What encoding does Chinese use?

Systems	Example (447)	Base	Base Dimension	Power Dimension
Abstract	$\Sigma a_i x^i$	x	a_i	x^i
Arabic	447	10	a_i = shape	x^i = position
	$4\times10^2+4\times10^1+7\times10^0$		0, 1, 2, ..., 9	... 10^2 10^1 10^0

一，而，三，四，五，六，七，八，九

十一，十二，十三，...

shape × shape

二十，二十一，二十二，...

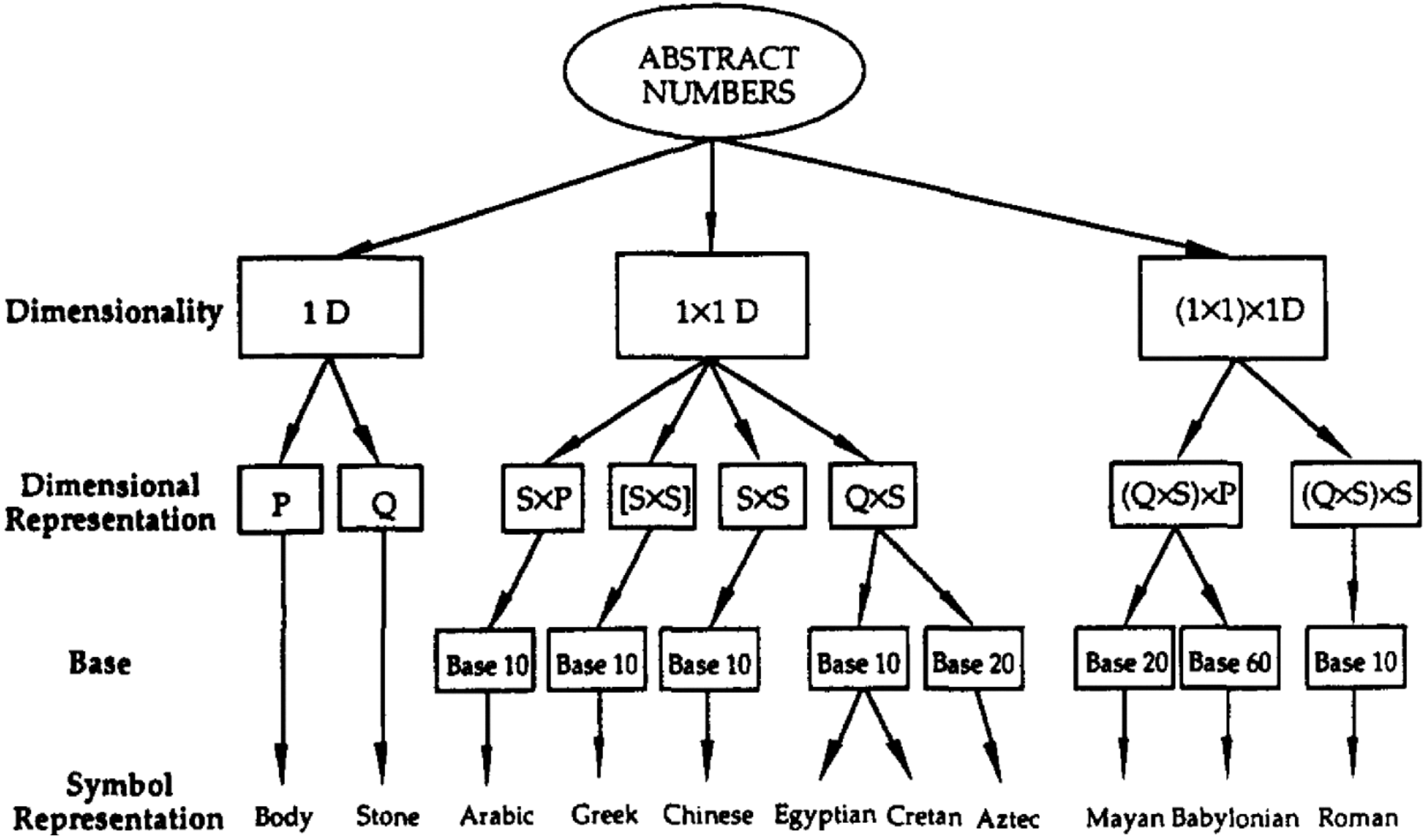
Zhang and Norman. "A Representational Analysis of Numeration Systems" 1995

Some numerals use sub-base as $(1 \times 1) \times 1D$

Systems	Example (447)	Main Base	Sub-base	Sub-base Dimension	Sub-power Dimension	Main Power Dimension
Abstract	$\Sigma\Sigma(b_{ij}y^j)x^i$	x	y	b_{ij}	y^j	x^i
Babylonian		60	10	b_{ij} = quantity	y^j = shape	x^i = position
	$(0\times10^1+7\times10^0)60^1$ $+(2\times10^1+7\times10^0)60^0$			Quantities of 𒁹's and 𒌋's	𒁹 $= 10^0$, 𒌋 $= 10^1$	... 60^2 60^1 60^0
Mayan		20*	5	b_{ij} = quantity	y^j = shape	x^i = position
	$(0\times5^1+1\times5^0)20^2$ $+(0\times5^1+2\times5^0)20^1$ $+(1\times5^1+2\times5^0)20^0$			Quantities of •'s and —'s.	• $= 5^0$, — $= 5^1$	... 20^2 20^1 20^0
Roman	CCCCXXXXVII	10	5	b_{ij} = quantity	y^j = shape	x^i = shape
	$(0\times5^1+4\times5^0)10^2$ $+(0\times5^1+4\times5^0)10^1$ $+(1\times5^1+2\times5^0)10^0$			Quantities of I's, V's, X's, L's, etc.	$I = 10^0\times5^0$ $V = 10^0\times5^1$ $X = 10^1\times5^0$ $L = 10^1\times5^1$ ...	$I = 5^0\times10^0$ $V = 5^1\times10^0$ $X = 5^0\times10^1$ $L = 5^1\times10^1$ ...

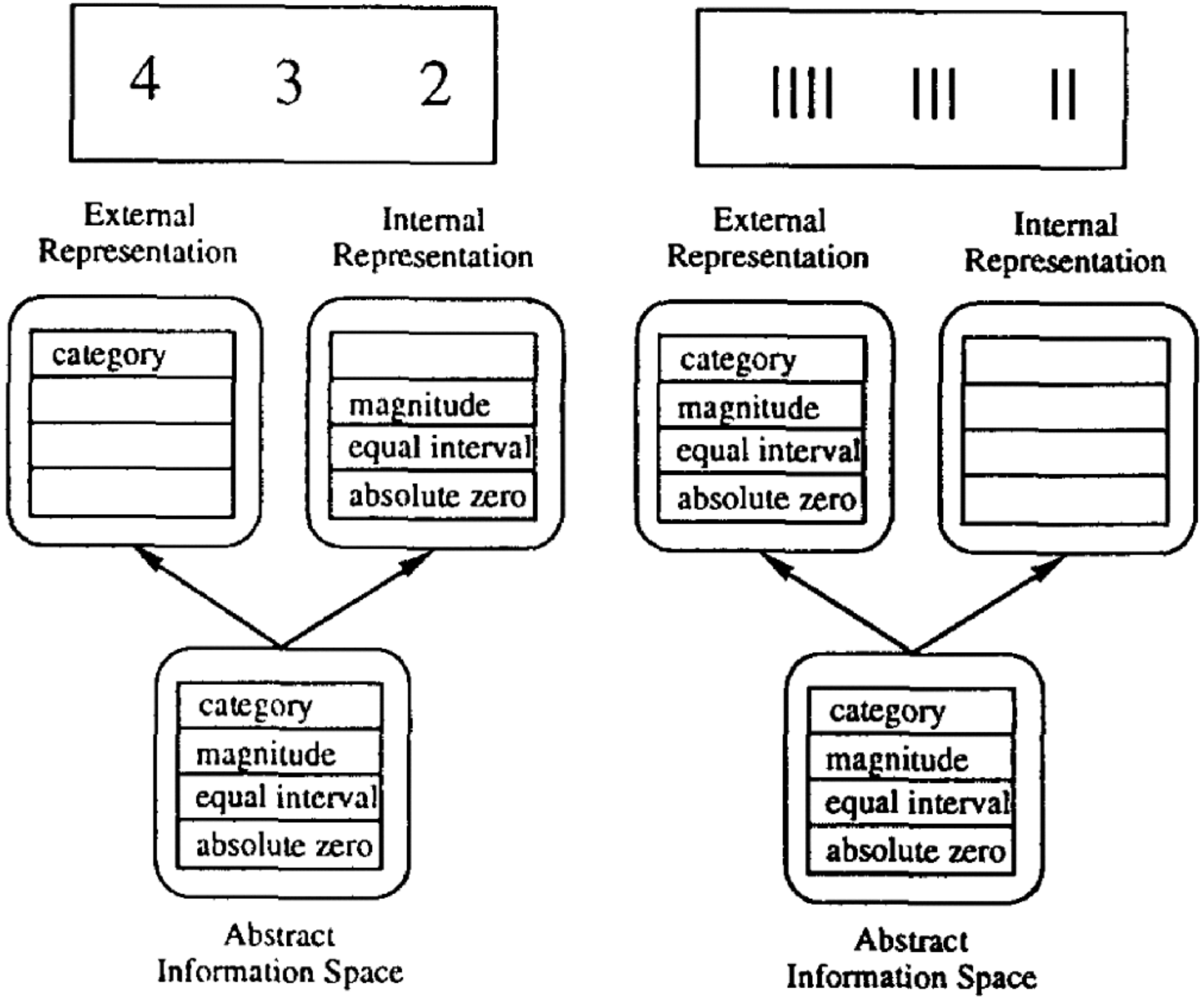
Zhang and Norman. "A Representational Analysis of Numeration Systems" 1995

Numerals form a notational design space



Zhang and Norman. "A Representational Analysis of Numeration Systems" 1995

Each notation supports different numeric tasks



(A) Arabic (B) Egyptian

Zhang and Norman. "A Representational Analysis of Numeration Systems" 1995

[long multiplication]

“So what makes the Arabic system so special? Numerals have two major functions: representation and calculation. In many cultures, these two functions are achieved by two separate systems. For example, in China, calculation was carried out by abacuses and sticks, whereas representation was realized by written numerals. [...]

We propose that what makes the Arabic system so special and widely accepted is that it integrates representation and calculation into a single system, in addition to its other nice features of efficient information encoding, compactness, extendibility, spatial representation, small base, effectiveness of calculation and, especially important, ease of writing.”

Zhang and Norman. “A Representational Analysis of Numeration Systems” 1995